Veto Interval Graphs

Stephanie Jones, Jessica Kawana, Dana Lapides
with Mentor Joshua D. Laison

Willamette Valley Mathematics Consortium REU-RET
Definition

A veto interval is a closed interval with three points: a left end point, a right end point, and a veto mark.
Definition

A \textit{veto interval representation} of a graph $G$ is a set of veto intervals such that two intervals intersect without containing any veto marks if and only if their corresponding vertices in $G$ are adjacent.
Theorem

The following families are Veto Interval Graphs

- Complete bipartite graphs
Theorem

The following families are Veto Interval Graphs

- Complete bipartite graphs
- Cycles with at least four vertices (can’t do three)
Introduction: Veto Interval Graphs

Theorem

The following families are Veto Interval Graphs

- Complete bipartite graphs
- Cycles with at least four vertices (can’t do three)
- Trees
Introduction: Veto Interval Graphs

Theorem

Not all subgraphs of VI graphs are VI graphs.
Definition

A *midpoint unit veto interval graph (MUVI)* is a veto interval graph with unit length intervals and midpoint veto marks.

Figure: Veto interval versus midpoint unit veto interval representation
Midpoint Unit Veto Interval Graphs

**Theorem**

The following families of graphs are MUVI

- Complete bipartite graphs
- Cycles with at least four vertices
- NOT all trees!
Which trees are MUVI?

Consider the following tree (5-lobster).
Which trees are MUVI?

Consider the following tree (5-lobster).
Which trees are MUVI?

Consider the following tree (5-lobster).
Which trees are MUVI?

Consider the following tree (5-lobster).
Which trees are MUVI?

Consider the following tree (5-lobster).
Which trees are MUVI?

Consider the following tree (5-lobster).
Which trees are MUVI?

Consider the following tree (5-lobster).
Which trees are MUVI?

Consider the following tree (5-lobster).
Which trees are MUVI?

Consider the following tree (5-lobster).

Theorem

*The 5-lobster is not MUVI.*
Consider the following tree (5-lobster).

Theorem

The 5-lobster is a veto interval graph.
Which trees are MUVI?

Is there a limit on the degree of the vertex?
Which trees are MUVI?

No! Because...

Theorem

All caterpillars are MUVI.
Which trees are MUVI?

Open Question
Which trees are MUVI?
The **chromatic number** of a graph is the smallest number of colors you need to color the graph so that any two vertices with an edge between them get different colors.
Triangle-free graphs

Triangle Free Graphs of High Chromatic Number

- Triangle-free graphs with minimal degree $\delta > \frac{2n}{5}$ are bipartite [Erdős and Holzman, 1994]
- If $\delta > \frac{n}{3}$, then the graph is 4-colorable [Brandt, 2002]
- Triangle-free graphs with $\delta > \frac{10n}{29}$ are 3-colorable [Jin, 1993]
Triangle-free graphs

Triangle Free Graphs of High Chromatic Number

- Triangle-free graphs with minimal degree $\delta > \frac{2n}{5}$ are bipartite [Erdős and Holzman, 1994]
- If $\delta > \frac{n}{3}$, then the graph is 4-colorable [Brandt, 2002]
- Triangle-free graphs with $\delta > \frac{10n}{29}$ are 3-colorable [Jin, 1993]

**Figure:** Grötzsch graph  Chvatal graph  Clebsch graph
Triangle-free graphs

Triangle Free Graphs of High Chromatic Number

- Triangle-free graphs with minimal degree $\delta > \frac{2n}{5}$ are bipartite [Erdős and Holzman, 1994]
- If $\delta > \frac{n}{3}$, then the graph is 4-colorable [Brandt, 2002]
- Triangle-free graphs with $\delta > \frac{10n}{29}$ are 3-colorable [Jin, 1993]

Figure: Grötzsch graph, Chvatal graph, Clebsch graph
Theorem

MUVI graphs are 3-colorable.
Theorem

MUVI graphs are 3-colorable.
MUVI Graphs are 3-Colorable

Theorem

MUVI graphs are 3-colorable.
Theorem

UVI graphs are 4-colorable.
UVI Graphs are 4-Colorable

Theorem

UVI graphs are 4-colorable.

-4  -3  -2  -1   0   1   2   3   4
UVI Graphs are 4-Colorable

Theorem
UVI graphs are 4-colorable.
Theorem
UVI graphs are 4-colorable.
Theorem

UVI graphs are 4-colorable.
Theorem
Proper midpoint VI graphs are 3-colorable.

Theorem
The following are minimal forbidden subgraphs:
- The Grötzsch graph
- The Chvatal graph

Theorem
There exists a family of VI graphs with unbounded greedy chromatic number.
There are seven distinct ways for two intervals with middle marks to interact.
Seven ways to interact

Type 1: Non intersecting intervals.
Seven ways to interact

Type 2: Veto interval type adjacency.
Seven ways to interact

1
2
3
4
5
6
7

Definition

A **single approval interval representation** of a graph $G$ is a set of veto intervals such that two intervals intersect with exactly one approval mark if and only if their corresponding vertices in $G$ are adjacent.

**Type 3 and 4: Single approval interval adjacency**
Seven ways to interact

Definition

A *double approval interval representation* of a graph $G$ is a set of veto intervals such that two intervals intersect with both approval marks if and only if their corresponding vertices in $G$ are adjacent.

Types 5, 6, and 7: Double approval interval type adjacency.
Adjacency for point-core bitolerance graphs, Golumbic and Trenk [2004], is the union of single approval and double approval type adjacency.
Given an interval representation, we can construct its corresponding graph.

An interval representation

An interval graph
Next we add a middle mark to each interval
Under the veto interpretation:

A veto interval graph
Under the single approval interpretation:

A single approval interval graph
Under the double approval interpretation:

A double approval interval graph
Under the point-core bitolerance interpretation:

A point-core bitolerance graph
Proposition

An interval graph can be partitioned into a veto interval graph and a point-core bitolerance graph with the same representation.
Proposition

A point-core bitolerance graph can be partitioned into a single approval interval graph and a double approval interval graph with the same representation.
Theorem

Interval graphs are properly contained in single approval interval graphs.

Theorem

Single approval interval graphs have arbitrarily large boxicity number.

Theorem

Midpoint unit double approval interval graphs are interval graphs.
The following families of graphs are single approval and double approval interval graphs.

1. Complete graphs.
2. Cycles.
3. Wheels.
4. Trees.
5. Complete bipartite graphs.

Complete \( k \)-partite graphs are single approval interval graphs.

Complete \( k \)-partite graphs are not double approval interval graphs.
We want to thank:
- The Willamette Valley Mathematics Consortium REU-RET
- The NSF for funding the consortium
- Our mentor, Dr. Joshua Laison


Any questions?