The Edge Coloring Game on Extended Stars

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Outline

1. Competitive Coloring
   - Basic Definitions
   - The Game
   - Extended Stars
   - Partial Extended Stars
   - Future Work
1 Competitive Coloring

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  - The Game
  - Extended Stars
  - Partial Extended Stars
  - Future Work
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Definitions

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The Set-Up

- A finite graph $G$
- A set $X$ of $r$ colors
- Two players: Alice and Bob
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Alice and Bob alternate coloring the uncolored edges of $G$.

At each step, each must use a legal color.

A color $\alpha$ in $X$ is legal for an uncolored edge $e$ if $e$ has no neighbors already colored $\alpha$. 
Game Time

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Competitive Graph Coloring
Winners and Losers

- Alice wins the game if all edges of $G$ are eventually colored.
- Bob wins if there comes a time in the game when there is an uncolored edge $e$ for which there is no legal color.
- The least $r$ such that Alice has a winning strategy is called the game chromatic index of $G$, denoted $\chi_g'(G)$. 
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\[ \chi_g'(G) \]
An Example of an edge coloring game

Figure: Beginning of the game
An Example of an edge coloring game

Figure: Alice's first turn.
An Example of an edge coloring game

Figure: Bob’s first turn.
An Example of an edge coloring game

Figure: Alice’s next turn.
An Example of an edge coloring game

Figure: Bob’s next turn.
An Example of an edge coloring game

Figure: Alice colors another edge.
An Example of an edge coloring game

Figure: We jump ahead, Bob plays.
Figure: Bob colors again, and Alice wins with $r = 5$. 
Cai and Zhu proved the following theorem:

**Theorem**

If $T$ is a tree, then $\chi_g'(T) \leq \Delta(T) + 2$.

This theorem brought up the question:

**Question**

Does there exist a tree such that $\chi_g'(T) = \Delta(T) + 2$?
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We say that a tree \( T \) is an extended star if \( \text{diam}(T) = 4 \), there exists a unique vertex \( v \in V(T) \) such that \( d(v) = \Delta(T) \), and for each \( u \in N(v) \), \( d(u) \geq 3 \).

- An internal edge is an edge that is incident with \( v \).
- An external edge is an edge that is not incident with \( v \).
The Game Chromatic Index

**Theorem**

If $T$ is an extended star, then $\chi_g'(T) = \Delta(T) + 1$.

To show that this is true we must present a strategy for Alice to win with $\Delta(T) + 1$ colors, and a strategy for Bob to win with $\Delta(T)$ colors.
The Game Chromatic Index

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If $T$ is an extended star, then $\chi'_g(T) = \Delta(T) + 1$.

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Our Extended Star
Alice’s Strategy

- We assume that $r = \Delta(T) + 1$.
- Alice will begin by coloring an internal edge.
- Alice will simply color an internal edge.
- If all internal edges have been colored, Alice will color an external edge.
Alice’s Strategy

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Playout For Alice’s Strategy
Alice colors first
Alice Colors
Bob Colors
Alice Colors
Bob Colors
Competitive Coloring

Alice Colors

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Competitive Graph Coloring
Now all that is left is to color the remaining external edges.
Bob’s Strategy

- We want to Alice and Bob to play with $r = \Delta(T)$.
- Alice will color first.
Bob’s Strategy

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- Alice will color first.
Bob’s Strategy

- If Alice colors on an external edge, and there are more than two uncolored internal edges left, Bob will color the incident internal edge.

Example
Bob’s Strategy

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Example
Bob’s Strategy

- If Bob follows the above strategy the game will eventually come to a point where there are two uncolored internal edges, $e$ and $f$ that are both incident with at least two uncolored external edges, $e_1, e_2, f_1, f_2$.

- Since $e$ and $f$ are the only two uncolored internal edges left, there will only be two legal colors for $e$ and $f$, we will say blue and green.
Alice colors $f$ with blue.
Bob colors $e_2$ with green.
Alice colors $f_2$ with an already used color.
Bob colors $e_2$ with blue.
If Alice colors $f$ with blue.
Bob will color $e_1$ with green winning.
What if Alice colors \( e \) green?
Bob will color $f_1$ blue.
As we have shown strategies for Bob and Alice we have proved our theorem.

**Theorem**

If $T$ is an extended star, then $\chi'_g(T) = \Delta(T) + 1$. 
As we have shown strategies for Bob and Alice we have proved our theorem.

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We say that $T$ is a **partial extended star** if $\text{diam}(T) = 4$, there exists a unique vertex $v \in V(T)$ such that $d(v) = \Delta(T)$, and for each $u \in N(v)$, $d(u) \geq 2$. 
We will call the vertices that are incident with exactly one external edge **weak vertices**, and all others, except the center, is a **strong vertices**.
Theorem

If $T$ is a partial extended star with fewer than $\lceil \Delta(T)/2 \rceil$ weak vertices, then $\chi_{g'}(T) = \Delta(T) + 1$.

Theorem

If $T$ is a partial extended star with more than $\lceil \Delta(T)/2 \rceil$ weak vertices, then the $\chi_{g'}(T) = \Delta(T)$.
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We wish to further study partial extended stars, mostly to attempt to prove the following conjecture.

**Conjecture**

Let $T$ be a partial extended star, if the number of weak vertices is equal to $\lfloor \Delta(T)/2 \rfloor$, then $\chi_g'(T) = \Delta(T)$.

We also still wish to find the case of a tree $T$ such that $\chi_g'(T) = \Delta(T) + 2$. 
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Let $T$ be a partial extended star, if the number of weak vertices is equal to $\left\lfloor \frac{\Delta(T)}{2} \right\rfloor$, then $\chi_g'(T) = \Delta(T)$.

We also still wish to find the case of a tree $T$ such that $\chi_g'(T) = \Delta(T) + 2$. 

References