The 2-Relaxed Game Chromatic Number of Complete Multipartite Graphs

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Notation and Recap

- $K_S$ is the complete multipartite graph with elements of $S = \{P_1, \ldots, P_n\}$ as partite sets.
- $K_{n\times r} = K_S d_i = r$ for $1 \leq i \leq n$.
- In the 2-relaxed game, a color $\alpha$ is legal for an uncolored vertex $v$, if every $\alpha$-colored vertex is adjacent to at most two other $\alpha$-colored vertices after $v$ is colored $\alpha$.
- If $G$ is a graph, $2 \chi_g(G)$ is the smallest $k$ such that Alice has a winning strategy for the 2-relaxed game on $G$ with $k$ colors.
Question: What is $2\chi_g(K_S)$, the 2-relaxed game chromatic number of $K_S$?

Answer:

- Difficult to determine in general.
- Progress limited to $K_{n^*r}$, for $r \leq 3$. 

Alex Sistko
Competitive Graph Coloring
**Question:** What is $2^{\chi_g}(K_S)$, the 2-relaxed game chromatic number of $K_S$?

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Lemma

Let \( \sigma = \sum_{k=1}^{n} d_i = |V(K_S)| \). Then \( 2^{\chi_g(K_S)} \leq \left\lceil \frac{\sigma}{3} \right\rceil \). If \( d_n \leq 4 \), \( 2^{\chi_g(K_S)} \geq \left\lceil \frac{\sigma}{4} \right\rceil \).

This (Almost) Means:

- Each color is used at least three times.
- If the partite sets are sufficiently small, each color is used at most four times.
- Caveat: sufficiently many uncolored vertices must exist.
Proof.

- Pick any three vertices:

  \[\lceil \frac{\sigma}{3} \rceil \text{ colors available} \Rightarrow 3 \lceil \frac{\sigma}{3} \rceil \geq \sigma \text{ vertices can be colored} \Rightarrow \text{Alice wins.}\]

- If \(d_n \leq 4\), pick any five vertices:

  \[\lceil \frac{\sigma}{4} \rceil - 1 \text{ colors available} \Rightarrow \text{at most } 4(\lceil \frac{\sigma}{4} \rceil - 1) < \sigma \text{ vertices can be colored} \Rightarrow \text{Bob wins.}\]
Theorem

Let $G = K_{n \times 1}$. Then $2^{\chi_g(G)} = \left\lceil \frac{n}{3} \right\rceil$.

Proof.

Trivial.
Theorem

Let $G = K_{n \times 1}$. Then $2\chi_g(G) = \lceil \frac{n}{3} \rceil$.

Proof.

Trivial.
Theorem

Let $G = K_{n \times 2}$. Then $2 \chi_g(G) = \lceil \frac{2n}{3} \rceil$.

**Strategy:** Each time Alice colors the first vertex of a partite set with an $\alpha$, Bob colors the other vertex in this partite set with some color $\beta$. He prefers to use a color which has not yet been played. If this is not possible, he uses a $\beta \neq \alpha$, and otherwise uses $\alpha$.

**Example (k=3):**

```
  O  O  O  O  O
  O  O  O  O  O
  O  O  O  O  O
```
Theorem

Let $G = K_{n\times 2}$. Then $2\chi_g(G) = \left\lceil \frac{2n}{3} \right\rceil$.

Strategy: Each time Alice colors the first vertex of a partite set with an $\alpha$, Bob colors the other vertex in this partite set with some color $\beta$. He prefers to use a color which has not yet been played. If this is not possible, he uses a $\beta \neq \alpha$, and otherwise uses $\alpha$.

Example (k=3):

- Alice colors the first vertex blue.
- Bob colors the other vertex white.
- Alice colors the next vertex blue.
- Bob colors the next vertex white.
- Alice colors the next vertex blue.
- Bob colors the next vertex white.
- Alice colors the next vertex blue.
- Bob colors the next vertex white.
- Alice colors the next vertex blue.
- Bob colors the next vertex white.
- Alice colors the next vertex blue.
- Bob colors the next vertex white.
- Alice colors the last vertex blue.
- Bob colors the last vertex white.

- The graph is colored with two colors.
Theorem

Let $G = K_{n\times 2}$. Then $2\chi_g(G) = \lceil \frac{2n}{3} \rceil$.

**Strategy:** Each time Alice colors the first vertex of a partite set with an $\alpha$, Bob colors the other vertex in this partite set with some color $\beta$. He prefers to use a color which has not yet been played. If this is not possible, he uses a $\beta \neq \alpha$, and otherwise uses $\alpha$.

**Example (k=3):**

![Graph Coloring Example](image-url)
Let $G = K_{n*2}$. Then $2\chi_g(G) = \lceil \frac{2n}{3} \rceil$.

**Strategy:** Each time Alice colors the first vertex of a partite set with an $\alpha$, Bob colors the other vertex in this partite set with some color $\beta$. He prefers to use a color which has not yet been played. If this is not possible, he uses a $\beta \neq \alpha$, and otherwise uses $\alpha$.

**Example (k=3):**

```
  +-------------------+
  |                  | 3
  |   +------------   +
  |   |            |   |
  |   |            |   |
  |   +------------+   
  |                  | 2
  +-------------------+
```
Theorem

Let $G = K_{n\times 2}$. Then $2\chi_g(G) = \lceil \frac{2n}{3} \rceil$.

**Strategy:** Each time Alice colors the first vertex of a partite set with an $\alpha$, Bob colors the other vertex in this partite set with some color $\beta$. He prefers to use a color which has not yet been played. If this is not possible, he uses a $\beta \neq \alpha$, and otherwise uses $\alpha$.

**Example (k=3):**

```
  Blue  Yellow  *  *  *
```

```
  Red   Blue  *  *  *
```
We want:

We have:
### Theorem

Let $G = K_{n \times 3}$. If $n$ is even, then $2\chi_g(G) = n$.

### Conjecture

Let $G = K_{n \times 3}$. If $n$ is odd, then $2\chi_g(G) \leq n - 1$.

### Future Work:
- Consider unequal partite set sizes.
- Increase size of partite sets.
We are an island of original thought.
Just Kidding: